Supersymmetric reduced models with a symmetry based on Filippov algebra

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Abstract

Generalizations of the reduced model of super Yang-Mills theory obtained by replacing the Lie algebra structure to Filippov n-algebra structures are studied. Conditions for the reduced model actions to be supersymmetric are examined. These models are related with what we call $\mathcal{N}_{min} = 2$ super p-brane actions.

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1 Introduction

Gauge symmetry based on Lie algebra [1] has a rather long history and it has successfully described weak and strong interactions in the nature. The non-Abelian Lie algebra gauge symmetry on the worldvolume of multiple D-branes was also a crucial ingredient in the recent developments in non-perturbative string theory. It was also essential in the matrix model proposals [2, 3] which use dimensionally reduced super Yang-Mills theory for definition.

Filippov n-algebra [4] is a natural generalization of Lie algebra. It began to attract wide attention from physicists recently after it appeared in a candidate model for multiple M2-branes [5, 6, 7, 8].

So far studies involving Filippov n-algebra in physics have been largely concentrated on the Filippov 3-algebra appearing in the multiple M2-brane model.[1] It will be interesting to look for other situations where Filippov n-algebra plays a role.

In this paper, we study generalizations of reduced super Yang-Mills theory obtained by replacing the Lie algebra structure to Filippov n-algebra, and examine when the reduced actions are supersymmetric. Reduced model is a candidate framework for a constructive definition of fundamental theory [3], and supersymmetry is expected to be a vital element in such a framework.

Another motivation for this study comes from a trial to relate the multiple M2-brane action with some covariant formalism, possibly the single M5-brane action [11, 12, 13, 14, 15] (see also [16]). In particular, Ref.[14] studied this issue from the viewpoint of space-time supersymmetry algebra. Although results in the above works suggest such a relation, complete understanding is still missing. In this paper, we will show that our reduced models have the same structure with a covariant Green-Schwarz type supermembrane action written in the membrane analogue of the Schild action [17]. This result will be a useful guide for understanding the above issue.

2 Filippov n-algebra

In this section we briefly review the necessary ingredients of Filippov n-algebra. The presentation in this section closely follows Ref.[18].

Filippov n-algebra [4], also known as n-Lie algebra, is a natural generalization of Lie algebra. (In this paper we will sometimes call it just n-algebra for short.) For a linear space $\mathcal{V} = \sum_{a=1}^{\dim \mathcal{V}} v_a T_a; v_a \in \mathbb{C}$, Filippov n-algebra structure is defined by a multi-linear

[1] With a notable exception of the Nambu bracket [9] which can be used to define a classic example of Filippov 3-algebra. Quantization of Nambu bracket and/or its application to brane models have been subjects of interests, see e.g. [10] and references therein.

map which we call n-bracket $[*, \cdots, *] : \mathcal{V}^{\otimes n} \to \mathcal{V}$ satisfying the following properties:

1. Skew-symmetry:
$$[A_{\sigma(1)}, \cdots, A_{\sigma(n)}] = (-1)^{|\sigma|}[A_1, \cdots, A_n]. \tag{2.1}$$

2. Fundamental identity:
$$\begin{aligned} & [A_1, \cdots, A_{n-1}, [B_1, \cdots, B_n]] \\ = & \sum_{k=1}^{n} [B_1, \cdots, B_{k-1}, [A_1, \cdots, A_{n-1}, B_k], B_{k+1}, \cdots, B_n]. \end{aligned} \tag{2.2}$$

In terms of the basis T_a, n-algebra is expressed in terms of the structure constants:
$$[T_{a_1}, \cdots, T_{a_n}] = i f_{a_1 \cdots a_n}{}^{b} T_b. \tag{2.3}$$

We introduce inner product as a bi-linear map $\mathcal{V} \times \mathcal{V} \to \mathbb{C}$:
$$\langle T_a, T_b \rangle = h_{ab}. \tag{2.4}$$

The symmetric tensor h_{ab} will be called metric of the n-algebra in the following.

We impose invariance of the metric
$$\langle [T_{a_1}, \cdots, T_{a_{n-1}}, T_b], T_c \rangle + \langle T_b, [T_{a_1}, \cdots, T_{a_{n-1}}, T_c] \rangle = 0. \tag{2.5}$$

This implies the tensor
$$f_{a_1 \cdots a_{n+1}} \equiv f_{a_1 \cdots a_n}{}^{b} h_{b a_{n+1}} \tag{2.6}$$

to be totally anti-symmetric.

We define Hermitian conjugation as follows:
$$[A_1, \cdots, A_n]^\dagger = [A_n^\dagger, \cdots, A_1^\dagger]. \tag{2.7}$$

3 Supersymmetric reduced model actions with a symmetry based on Filippov n-algebra

IIB matrix model [3] is defined as a large N reduced model of ten dimensional super Yang-Mills theory. Its action is given by
$$S = \frac{1}{4}\langle [X_I, X_J], [X^I, X^J] \rangle + \frac{1}{2}\langle \bar{\Psi}, \Gamma_I [X^I, \Psi] \rangle. \tag{3.1}$$

Here, X^I ($I = 1, \cdots, 10$) is a vector in ten dimensional flat target space-time and Ψ is a space-time Majorana-Weyl spinor, both take values in $U(N)$ Lie-algebra. Γ_I's are

gamma matrices in ten dimension. Repeated vector indices are contracted by space-time metric $\eta_{IJ} = \mathrm{diag}(+, -, \cdots, -)$. We have used the Filippov n-algebra notations ($n = 2$ for ordinary Lie algebra) described in the previous section. The inner product is given by the invariant trace of the Lie algebra.

The action (3.1) is invariant under the following supersymmetry transformation:

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi, \\ \delta\Psi &= \frac{i}{2}[X^I, X^J]\Gamma_{IJ}\epsilon. \end{aligned} \tag{3.2}$$

A natural generalization of the action (3.1) based on Filippov $(p+1)$-algebra would be

$$\begin{aligned} S = &\frac{1}{2(p+1)!}\langle [X_{I_1}, \cdots, X_{I_{p+1}}][X^{I_1}, \cdots, X^{I_{p+1}}]\rangle \\ &+\frac{\sigma}{2}\langle \bar{\Psi}, \Gamma_{I_1\cdots I_p}[X^{I_1}, \cdots, X^{I_p}, \Psi]\rangle. \end{aligned} \tag{3.3}$$

Here σ is a factor 1 or i determined from the Hermiticity of the action. X^I ($I = 1, \cdots, D$) is a vector in D-dimensional flat target space-time and Ψ is a space-time spinor, both take value in $(p+1)$-algebra. Γ_I's are D dimensional gamma matrices satisfying

$$\Gamma_I\Gamma_J + \Gamma_J\Gamma_I = 2\eta_{IJ}, \tag{3.4}$$

where η_{IJ} is now D-dimensional flat metric with $\eta_{IJ} = \mathrm{diag}(\overbrace{+, \cdots, +}^{t}, \overbrace{-, \cdots, -}^{s})$. We allow the number of the time-like directions to be general t. $\Gamma_{I_1\cdots I_p}$ is an anti-symmetrized product of gamma matrices with "strength one".

The action (3.3) is invariant under a transformation

$$\delta\Phi^a = \Lambda^{a_1\cdots a_p} f_{a_1\cdots a_p b}{}^a \Phi^b, \quad \Phi^a = X^{Ia}, \Psi^a, \tag{3.5}$$

due to the fundamental identity (2.2) and the invariance of the inner product (2.5). This is a natural generalization of the dimensionally reduced gauge symmetry of the action (3.1).

In this paper, we examine in which case the following supersymmetry transformation

$$\begin{aligned} \delta X^I &= c_1\bar{\epsilon}\Gamma^I\Psi, \\ \delta\Psi &= c_2[X^{I_1}, \cdots, X^{I_{p+1}}]\Gamma_{I_1\cdots I_{p+1}}\epsilon, \end{aligned} \tag{3.6}$$

leaves the action (3.3) invariant. Here, c_1 and c_2 are coefficients to be adjusted. We will keep the Filippov algebra to be general, i.e. we will not use any property specific to a particular Filippov algebra. The conditions we may impose on fermions are the

standard ones, i.e. (pseudo-)Majorana condition and Weyl condition. We will not consider projections on fermions which break the $SO(t,s)$ Lorentz symmetry. In order for the second term in the action (3.3) to be not identically zero, when we impose Weyl condition on fermions $t+p$ must be even, and when fermions are Majorana-spinors $\Gamma_{I_1\cdots I_p}C$ must be symmetric in spinor indices. Here, C is the charge conjugation matrix. The properties of gamma matrices and spinors in diverse dimensions are summarized in appendix A.

Let us first study the variation of the action which has one fermion. The variation of the second term in the action (3.3) containing one fermion has a form

$$\begin{aligned} & \langle\bar{\epsilon}\Gamma^{J_1\cdots J_{p+1}}\Gamma^{I_1\cdots I_p}[X_{J_1},\cdots,X_{J_{p+1}}][X_{I_1},\cdots,X_{I_p},\Psi]\rangle \\ = \;& -\langle\bar{\epsilon}\Gamma^{J_1\cdots J_{p+1}}\Gamma^{I_1\cdots I_p}[X_{I_1},\cdots,X_{I_p},[X_{J_1},\cdots,X_{J_{p+1}}]]\Psi\rangle, \end{aligned} \tag{3.7}$$

where we have used the invariance of the inner product (2.5). One can rearrange the ordering of the gamma matrices into a sum of totally anti-symmetrized gamma matrices using (3.4):

$$\begin{aligned} \Gamma_{J_1\cdots J_{p+1}}\Gamma^{I_1\cdots I_p} \;=\;& \Gamma_{J_1\cdots J_{p+1}}{}^{I_1\cdots I_p} \\ &+ (-)^p\delta^{[I_1}_{[J_1}\Gamma_{J_2\cdots J_{p+1}]}{}^{I_2\cdots I_p]} \\ &+ \delta^{[I_1}_{[J_1}\delta^{I_2}_{J_2}\Gamma_{J_2\cdots J_{p+1}]}{}^{I_3\cdots I_p]} \\ &+ \cdots \\ &+ (-)^p\delta^{[I_1}_{[J_1}\cdots\delta^{I_p]}_{J_p}\Gamma_{J_{p+1}]}, \end{aligned} \tag{3.8}$$

where the square brackets on Lorentz indices denote total anti-symmetrization with appropriate "strength" (it will be relevant only for the last term). On the other hand, using the fundamental identity (2.2) one can show

$$\Gamma^{I_1\cdots I_r J_1\cdots J_{r+1}}[A_1,\cdots,A_{p-r},X_{I_1},\cdots,X_{I_r},[A_1,\cdots,A_{p-r},X_{J_1},\cdots,X_{J_{r+1}}]]=0, \tag{3.9}$$

for $r\neq 0$, with pairs of the same entries $A_1,\cdots,A_{p-r}$. Due to (3.9) the terms from (3.7) arising from the rearrangement of the gamma matrices (3.8) mostly vanish; only the $r=0$ term remains which cancels the similar term coming from the variation of the first term in the action (3.3).

Next, let us examine the variation of the action containing three fermions. Since the structure constant $f_{a_1\cdots a_{p+1}}{}^b$ is anti-symmetric in indices $a_1\cdots a_{p+1}$ due to the skew-symmetric property (2.1), the variation containing three fermions vanishes when

$$(\Gamma_I)^\alpha{}_\beta(\Gamma^{IJ_1\cdots J_{p-1}})^\gamma{}_\delta\Psi^{[a_1}_\beta\bar{\Psi}^{a_2}_\gamma\Psi^{a_3]}_\delta=0, \tag{3.10}$$

where the square bracket denotes the total anti-symmetrization in the $(p+1)$-algebra indices. (3.10) is equivalent to

$$(\Gamma_I P)^\alpha{}_{(\beta}(\Gamma^{IJ_1\cdots J_{p-1}}P)^\gamma{}_{\delta)}=0, \tag{3.11}$$

when Ψ's are complex spinors, where P is a chiral projection when Ψ's are Weyl spinors and 1 otherwise, and

$$(\Gamma_I CP)_{\alpha(\beta}(\Gamma^{IJ_1\cdots J_{p-1}}CP)_{\gamma\delta)}=0, \tag{3.12}$$

when Ψ's are (pseudo-)Majorana(-Weyl) spinors. From an argument similar to the one in [19], when (3.10) is satisfied it follows that

$$D-p-1=\frac{1}{2}n_f, \tag{3.13}$$

where n_f is the spinor size of the fermions Ψ counted in the real number.[2] We provide a proof in the appendix B. One can also check that (3.13) is a sufficient condition for (3.10) to vanish by expanding the left hand side of (3.11) or (3.12) by complete basis of matrices with indices α and β. We list the cases with $p \geq 2$ when (3.10) is satisfied in Table 1 ($p=1$ case is the ordinary Lie-algebra case which can be easily included). The columns for θ and n_θ in the table are about corresponding super p-branes which we will discuss in section 5.

D	t	p	Ψ	n_f	θ	n_θ
4	2	2	(pseudo-)Majorana-Weyl	2	(pseudo-)Majorana	4
5	2	2	Majorana	4	Dirac	8
5	3	2	pseudo-Majorana	4	Dirac	8
6	3	3	(pseudo-)Majorana-Weyl	4	Weyl	8

Table 1: The dimension of the target space-time D and the number of its time-like dimensions t where the supersymmetric reduced model with $(p+1)$-algebra symmetry and corresponding Green-Schwarz type super p-brane exist. The column under Ψ is the spinor property of the fermions in the supersymmetric reduced models and n_f is the spinor size of Ψ, and the column under θ is the spinor property of the space-time spinor fields θ of the corresponding Green-Schwarz type super p-branes and n_θ is its spinor size.

We explicitly write down the supersymmetric reduced model action in the case of $D=4$, $t=2$, $p=2$ with pseudo-Majorana-Weyl conditions on fermions:

$$S=\frac{1}{6}\langle[X_I,X_J,X_K][X^I,X^J,X^K]\rangle+\frac{1}{2}\langle\bar{\Psi}\Gamma_{IJ}[X^I,X^J,\Psi]\rangle. \tag{3.14}$$

[2] When Dirac spinors have $n_D^{\mathbb{C}}=\frac{1}{2}n_D^{\mathbb{R}}$ spinor components, by the size of the spinor counted in real number we mean $n_D^{\mathbb{R}}$. (pseudo-)Majorana condition or Weyl condition reduces size of spinors by half: (pseudo-)Majorana spinors and Weyl spinors have size $\frac{1}{2}n_D^{\mathbb{R}}$ and (pseudo-)Majorana-Weyl spinors have size $\frac{1}{4}n_D^{\mathbb{R}}$.

The supersymmetry transformation is given by

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi, \\ \delta\Psi &= \frac{i}{6}[X^I, X^J, X^K]\Gamma_{IJK}\epsilon. \end{aligned} \tag{3.15}$$

The case with Majorana-Weyl fermions is similar, with appropriate modifications in the coefficients in (3.15).[3]

4 Super Poincaré algebra

In the previous section we called the fermionic transformation (3.15) "supersymmetry transformation", since it is an analogue of the supersymmetry of the reduced model of super Yang-Mills theory. However, we haven't shown its relation to the standard supersymmetry algebra, namely super Poincaré algebra. Let us examine this point in this section.

We again take the case $D = 4$, $t = 2$, $p = 2$ for explicitly. Other cases are similar. In (3.14), the fermions are pseudo-Majorana-Weyl spinors:

$$C\bar{\Psi}^T = \Psi, \quad P_+\Psi = \Psi, \tag{4.1}$$

where

$$P_\pm \equiv \frac{1 \pm \Gamma_5}{2}, \quad \Gamma_5 \equiv \Gamma_1\Gamma_2\Gamma_3\Gamma_4. \tag{4.2}$$

It is important to notice that when the 3-algebra has a central element, there is a fermionic shift symmetry:[4]

$$\delta_+ X^I = 0, \quad \delta_+\Psi^a = \delta^{a\odot}\epsilon_+, \tag{4.3}$$

where $\odot$ denotes the central element: $[T_\odot, T_a, T_b] = 0$ for $^\forall T_a, T_b$. The commutation relations of the two fermionic transformations turn out to be

$$\begin{aligned} (\delta_+(\epsilon_+^{(1)})\delta_+(\epsilon_+^{(2)}) - (1 \leftrightarrow 2))\Phi &= 0, \quad \Phi = X^I, \Psi, && (4.4) \\ (\delta_+(\epsilon_+)\delta_-(\epsilon_-) - \delta_-(\epsilon_-)\delta_+(\epsilon_+))X^{Ia} &= \delta^{a\odot} i\bar{\epsilon}_-\Gamma^I\epsilon_+, && (4.5) \\ (\delta_+(\epsilon_+)\delta_-(\epsilon_-) - \delta_-(\epsilon_-)\delta_+(\epsilon_+))\Psi &= 0, \\ (\delta_-(\epsilon_-^{(1)})\delta_-(\epsilon_-^{(2)}) - (1 \leftrightarrow 2))\Phi_a &= \Lambda_a{}^b\Phi_b, && (4.6) \end{aligned}$$

[3]Actually in the action (3.14), the difference between Majorana-Weyl fermions and pseudo-Majorana fermions is just a matter of convention: The difference arises from the choice of the charge conjugation matrix $C_{\eta'=1}$ (for Majorana fermions) or $C_{\eta'=-1}$ (for pseudo-Majorana fermions) in (A.2) in the appendix A, which are related as $C_{\eta'=1} = \Gamma_5 C_{\eta'=-1}$. Using the Weyl condition on fermions, one can see that the action for pseudo-Majorana-Weyl fermions and that for Majorana-Weyl fermions are exactly the same.

[4]The role of the fermionic shift symmetry in the multiple M2-brane model was studied extensively in [14].

where

$$\Lambda_a{}^b = i f^{bcd}{}_a X_c^K X_d^L \bar{\epsilon}_-^{(1)} \Gamma_{KL} \epsilon_-^{(2)}. \tag{4.7}$$

Here, $\delta_+(\epsilon_+)$ and $\delta_-(\epsilon_-)$ denote the fermionic shift (4.3) with parameter ϵ_+ and supersymmetry transformation (3.6) with parameter ϵ_-, respectively. (4.5) is a translation in the target space-time. Thus the supersymmetry transformation (3.15) together with the fermionic shift (4.3) form target space-time super Poincaré algebra, modulo the right hand side of (4.6) which has a form of the symmetry transformation (3.5). We will use $\mathcal{N}_{min}$ to count the number of supersymmetry in the unit of the minimal spinor. In this notation, our model has $\mathcal{N}_{min} = 2$ space-time supersymmetry when there is a central element in the algebra, since the minimal spinor in $D = 4$, $t = 2$ is (pseudo-)Majorana-Weyl spinor. However, as can be seen from (4.5) it is not possible to construct $\mathcal{N}_{min} = 1$ space-time super Poincaré algebra in four dimension by using just one minimal spinor. In this sense $\mathcal{N}_{min} = 2$ supersymmetry is minimal in $D = 4$, $t = 2$ and hence it is what should be called $\mathcal{N} = 1$ supersymmetry.

So far we have been studying super Poincaré algebra in four dimension. However, the supersymmetry transformation (3.15) can form super Poincaré algebra in three dimension when particular background is chosen. As an example, let us choose the 3-algebra to be Nambu-Poisson bracket in $R^{2,1}$:

$$[f(y), g(y), h(y)] = i\epsilon^{ijk} \partial_i f(y) \partial_j g(y) \partial_k h(y), \tag{4.8}$$

$$\langle f(y), g(y) \rangle = \int d^3y f(y) g(y), \tag{4.9}$$

where y^i $(i = 1, 2, 3)$ are flat coordinates on $R^{2,1}$ and ϵ^{ijk} is the Levi-Civita symbol. We consider following background configuration:

$$\begin{aligned} X^I(y) &= y^I \quad (I = 1, 2, 3), \\ X^4(y) &= 0. \end{aligned} \tag{4.10}$$

Then, (4.7) becomes

$$(\delta_-(\epsilon_-^{(1)})\delta_-(\epsilon_-^{(2)}) - (1 \leftrightarrow 2))\tilde{\Phi} \sim \epsilon^{ijk} \bar{\epsilon}_-^{(1)} \Gamma_{jk} \epsilon_-^{(2)} \partial_i \tilde{\Phi} + \cdots, \tag{4.11}$$

where $\tilde{\Phi}$ are fluctuation of the fields around the background (4.10). To see this is a super Poincaré algebra in three dimension, one decomposes gamma matrices and supersymmetry transformation parameters to those for three dimension. Then (4.11) can be rewritten as

$$(\delta_-(\zeta^{(1)})\delta_-(\zeta^{(2)}) - (1 \leftrightarrow 2))\tilde{\Phi} \sim \bar{\zeta}^{(1)} \gamma^i \zeta^{(2)} \partial_i \tilde{\Phi} + \cdots, \tag{4.12}$$

where $\cdots$ can be combined into a form of gauge transformation [12] and γ^i $i = 1, 2, 3$ and ζ are gamma matrices and supersymmetry transformation parameters in three dimension. To keep $\tilde{\Phi} = 0$ configuration to preserve supersymmetry, one also needs to combine the fermionic shift (4.3) [12]. Thus in the background (4.10) the supersymmetry transformation (3.15) appropriately combined with the fermionic shift (4.3) can be regarded as super Poincaré symmetry in three dimension.

5 Relation to $\mathcal{N}_{min} = 2$ super p-branes

In this section we show that our supersymmetric reduced model actions can be related to Green-Schwarz type $\mathcal{N}_{min} = 2$ super p-action in the Schild-type form, parallel to the relation between the large N reduced model action of super Yang-Mills theory and Green-Schwarz superstring action [3].[5] To be explicit, we again take $D = 4$, $t = 2$, $p = 2$ case as an example. Discussions are parallel in other cases listed in Table 1.

The super p-brane action with $p = 2$, i.e. the supermembrane action is given by

$$S_{GS} = \int d^3y \left(\frac{1}{2}\sqrt{-g}g^{ij}E_i^I E_j^J \eta_{IJ} - \frac{1}{2}\sqrt{-g} + \epsilon^{ijk}E_i^A E_j^B E_k^C B_{CBA} \right), \tag{5.1}$$

where we take the worldvolume signature as $(++-)$ and $A = (I, \alpha)$, and

$$E_i^I = \partial_i X^I - \frac{i}{2}\bar{\theta}\Gamma^I \partial_i \theta, \quad E_i^\alpha = \partial_i \theta^\alpha. \tag{5.2}$$

Here, θ is a pseudo-Majorana spinor in $D = 4$, $t = 2$ target space-time:

$$C\bar{\theta}^T = \theta. \tag{5.3}$$

B_{ABC} is determined from $dB = H$ and $dH = 0$, where

$$\begin{aligned} B &= \frac{1}{3!}E^A E^B E^C B_{ABC}, \\ H &= \frac{1}{4!}E^A E^B E^C E^D H_{ABCD}, \quad E^A = E_i^A dy^i, \end{aligned} \tag{5.4}$$

and the only non-zero components of H_{ABCD} are those with two spinor and two vector indices:

$$H_{\alpha\beta IJ} = -\frac{i}{6}(C^{-1T}\Gamma_{IJ})_{\alpha\beta}. \tag{5.5}$$

[5] Green-Schwarz type supermembrane actions with general space-time signatures have been studied in [20]. However, there study was restricted to $\mathcal{N}_{min} = 1$ case in our terminology, and $\mathcal{N}_{min} = 2$ supersymmetry which we discuss in this paper was not considered there.

The closure of H is equivalent to the identity

$$(\Gamma_I C)_{(\alpha\beta}\Gamma^{IJ}C_{\gamma\delta)} = 0. \tag{5.6}$$

From this condition one obtains the matching of on-shell degrees of freedom between bosons and fermions when $2 < p+1 < d$ [19]:

$$D - (p+1) = \frac{1}{4} n_{min}\mathcal{N}_{min}, \tag{5.7}$$

where n_{min} is the dimension of the minimal spinor. (5.7) is satisfied for $p = 2$, $D = 4$, $\mathcal{N}_{min} = 2$ with pseudo-Majorana-Weyl spinor as the minimal spinor; $n_{min} = 2$. Indeed, one can show that (5.6) is satisfied in this case.

The action (5.1) is invariant under the following global space-time supersymmetry transformation:

$$\delta X^I = \frac{i}{2}\bar{\epsilon}\Gamma^I\theta, \quad \delta\theta = \epsilon. \tag{5.8}$$

In terms of the minimal spinor, the action has $\mathcal{N}_{min} = 2$ non-chiral space-time supersymmetry.

The action (5.1) also has the local fermionic gauge symmetry:

$$\delta X^I = \frac{i}{2}\bar{\theta}\Gamma^I(1+\Gamma)\kappa, \quad \delta\theta = (1+\Gamma)\kappa, \tag{5.9}$$

where

$$\Gamma \equiv \frac{1}{3!\sqrt{-g}}\epsilon^{ijk}E^I_i E^J_j E^K_k \Gamma_{IJK}. \tag{5.10}$$

The transformation law for the worldvolume metric g_{ij} can be determined as in [21]. To relate the supermembrane action with our reduced model action (3.14), we fix the fermionic gauge symmetry by the condition[6]

$$P_-\theta = 0, \tag{5.11}$$

where

$$P_\pm \equiv \frac{1}{2}(1 \pm \Gamma_5), \quad \Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4. \tag{5.12}$$

The supersymmetry transformation must be combined with the global part of the fermionic gauge transformation to maintain the gauge condition (5.11). Then the supersymmetry transformation becomes

$$\delta X^I = i\bar{\epsilon}_-\Gamma^I\Psi, \tag{5.13}$$

[6]This gauge condition is appropriate for configurations which break the part of the supersymmetry generated by ϵ_+.

$$\delta\Psi = \epsilon_+ - \Gamma\epsilon_-, \tag{5.14}$$

where $\Psi = P_+\theta$. After the gauge fixing (5.11), the action takes the form

$$S_{GS} = \int d^3y \left(\frac{1}{2}\sqrt{-g}g^{ij}\partial_i X^I \partial_j X^J \eta_{IJ} - \frac{1}{2}\sqrt{-g} - \frac{i}{4}\epsilon^{ijk}\partial_i X^I \partial_j X^J \bar{\Psi}\Gamma_{IJ}\partial_k\Psi\right). \tag{5.15}$$

This action is classically equivalent to the following Schild type action:

$$\begin{aligned} S_{Schild} &= \frac{1}{2}\int d^3y\, w(y) \\ &\quad \left(-\frac{1}{6}\{X^I, X^J, X^K\}\{X_I, X_J, X_K\} - \frac{1}{2}\bar{\Psi}\Gamma^{IJ}\{X_I, X_J, \Psi\} + 1\right), \end{aligned} \tag{5.16}$$

where $w(y)$ is identified with the volume density and $\{*, *, *\}$ is the Nambu-Poisson bracket

$$\{f, g, h\} \equiv \frac{i}{w(y)}\epsilon^{ijk}\partial_i f \partial_j g \partial_k h. \tag{5.17}$$

The action (5.16) can be identified with our supersymmetric reduced model (3.14) with 3-algebra being the Nambu-Poisson bracket (5.17). The supersymmetry transformation now becomes

$$\delta X^I = i\bar{\epsilon}_-\Gamma^I\Psi, \tag{5.18}$$

$$\delta\Psi = \epsilon_+ + \frac{i}{6}\{X^I, X^J, X^K\}\Gamma_{IJK}\epsilon_-. \tag{5.19}$$

The fermionic shift symmetry with the parameter ϵ_+ is identical to (4.3), and the supersymmetry transformation parametrized by ϵ_- is identical to (3.6).

Similar discussions go through in other cases listed in the Table 1. $D = 5$ models are related with $D = 6$ model by a (formal) double dimensional reduction. In all cases n_θ is the minimal size of the spinor needed to have super Poincaré algebra and it is twice as big as the size of the minimal spinor n_{min} in that space-time dimension and signature. Thus all super p-brane actions in the Table 1 have $\mathcal{N} = 1$, $\mathcal{N}_{min} = 2$ target space supersymmetry. Note that the condition for the existence of the Green-Schwarz type super p-branes (5.7) coincides with the condition for the existence of the supersymmetric reduced models (3.13) with $n_f = n_{min}$ and $\mathcal{N}_{min} = 2$, as it should be.

6 Summary and future directions

In this paper, we constructed supersymmetric reduced model actions with a symmetry based on Filippov algebra. These models are natural generalizations of the reduced model

of super Yang-Mills theory. The supersymmetry transformation itself involves the Filippov algebra structure, and our models compactly exhibit interrelation between supersymmetry and the Filippov algebra symmetry.

The supersymmetric reduced models were related with what we call $\mathcal{N}_{min} = 2$ super p-brane actions. In rewriting the super p-brane actions in the form of our reduced models, there was no truncation of the terms of the super p-brane actions. Since our models capture the aspects of the symmetries in a compact form, they will provide a good guidance for the issue of relating the multiple M2-brane model with some covariant formalism [11, 12, 13, 14, 15]. Our models have a nice feature that the D-dimensional Lorentz covariance is manifest. This was due to $\mathcal{N}_{min} = 2$ supersymmetry which allowed us to fix the fermionic gauge symmetry in the Lorentz covariant form (5.11).[7] This is in contrast to the multiple M2-brane model or super p-branes in the light-cone gauge which have similar algebraic structures [23, 24], and may become an advantage for understanding the structure of the space-time at more fundamental level. In particular, it will be useful for describing space-time uncertainty principle covariantly [25].

One of the advantage of reduced models is that the path integral reduces to ordinary integral and sometimes explicit integration is possible, e.g. [26, 27, 28]. Together with the highly symmetric nature of our models, we may be able to perform the path integral explicitly and learn quantum aspects of the models with those symmetries.

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A Gamma matrices and spinors in diverse dimensions

In this appendix we summarize the properties of gamma matrices and spinors in diverse dimensions. See [29] for more detail.

[7] While we were completing this work, a paper [22] appeared which proposed a reduced model with Filippov 3-algebra structure as a covariant formulation of M-theory. Only the bosonic part was constructed in that paper. Since M-theory has $\mathcal{N} = \mathcal{N}_{min} = 1$ supersymmetry of eleven dimensional space-time as opposed to our $\mathcal{N}_{min} = 2$ models, one cannot follow our approach in their model. To keep the covariance under eleven dimensional Lorentz transformation in their model when including terms for supersymmetric completion, one would need to modify the structure of the model considerably.

D dimensional gamma matrices Γ_I $(I = 1, \cdots, D)$ satisfy

$$\Gamma_I \Gamma_J + \Gamma_J \Gamma_I = 2\eta_{IJ}, \tag{A.1}$$

where η_{IJ} is D-dimensional flat metric with $\eta_{IJ} = \mathrm{diag}(\overbrace{+, \cdots, +}^{t}, \overbrace{-, \cdots, -}^{s})$.

$$\Gamma_I^\dagger = \begin{cases} \Gamma_I & (I = 1, \cdots, t) \\ -\Gamma_I & (I = t+1, \cdots, D). \end{cases}$$

The charge conjugation matrix C is characterized by the property

$$C^{-1} \Gamma_I C = \eta' \Gamma_I^T \quad (\eta' = \pm 1). \tag{A.2}$$

For even D either sign of η' can be chosen, but for odd D it is fixed to $\eta' = \frac{D(D-1)}{2}$.

$$C^T = \varepsilon' C, \quad C^\dagger C = 1, \tag{A.3}$$

it follows that

$$\Gamma_{I_1 \cdots I_r} C = (\eta')^r \varepsilon' (-)^{\frac{r(r-1)}{2}} (\Gamma_{I_1 \cdots I_r} C)^T, \tag{A.4}$$

where $\Gamma_{I_1 \cdots I_r}$ is totally anti-symmetrized product of gamma matrices with "strength one". The rank r of symmetric and anti-symmetric $\Gamma_{I_1 \cdots I_r} C$ for the relevant cases are listed in Table 2.

To define Dirac conjugation, we introduce

$$\Gamma_0 \equiv \Gamma_1 \cdots \Gamma_t. \tag{A.5}$$

It satisfies

$$\Gamma_0 \Gamma_0^\dagger = 1, \quad \Gamma_0^\dagger = (-)^{\frac{t(t-1)}{2}} \Gamma_0, \tag{A.6}$$

$$\Gamma_0 \Gamma_I^\dagger \Gamma_0^\dagger = (-)^{t+1} \Gamma_I. \tag{A.7}$$

Dirac conjugate field $\bar{\psi}$ of ψ is defined as

$$\bar{\psi} = \psi^\dagger \Gamma_0^{-1}. \tag{A.8}$$

Let us introduce following matrix B:

$$\begin{aligned} &B^{-1} \Gamma_I B = \eta \Gamma_I^* \quad (\eta = \pm 1), \\ &B^T = \varepsilon B, \quad B^\dagger B = 1. \end{aligned} \tag{A.9}$$

Charge conjugate field ψ^c of ψ is defined by

$$\psi^c = C\bar{\psi}^T = B\psi^*. \tag{A.10}$$

It follows that

$$B = C\Gamma_0^*, \tag{A.11}$$

and

$$\eta = \eta'(-)^{t+1}, \quad \varepsilon = \varepsilon'(\eta')^t(-)^{\frac{t(t-1)}{2}}. \tag{A.12}$$

(pseudo-)Majorana spinors ψ_M satisfy

$$\psi_M = C\bar{\psi}_M^T = B\psi_M^*. \tag{A.13}$$

This is possible only when $\varepsilon = +1$ since (A.13) implies $BB^* = 1$.

When D is even, we can define Γ_{D+1} as

$$\Gamma_{D+1} = i^{(s-t)/2}\Gamma_1\Gamma_2\cdots\Gamma_D, \tag{A.14}$$

which satisfies

$$\Gamma_{D+1}^2 = 1, \quad \Gamma_{D+1}^\dagger = \Gamma_{D+1}. \tag{A.15}$$

Γ_{D+1} has following B-conjugation property:

$$B^{-1}\Gamma_{D+1}B = (-)^{(s-t)/2}\Gamma_{D+1}^* \tag{A.16}$$

Weyl spinors $\psi_\pm$ satisfy

$$\Gamma_{D+1}\psi_\pm = \pm\psi. \tag{A.17}$$

However this is compatible with the (pseudo-)Majorana condition (A.13) only if

$$(-)^{(s-t)/2} = 1, \tag{A.18}$$

i.e. $s - t = 0$ mod 4. The values of $s - t$ when (pseudo-)Majorana(-Weyl) spinors exist are listed in Table 3.

D	η'	ε'	r of symmetric $\Gamma_{I_1\cdots I_r}C$	r of anti-symmetric $\Gamma_{I_1\cdots I_r}C$
4	$+$	$-$	2,3	0,1,4
	$-$	$-$	1,2	0,3,4
5	$+$	$-$	2,3	0,1,4
6	$+$	$-$	2,3,6	0,1,4,5
	$-$	$+$	0,3,4	1,2,5,6
7	$-$	$+$	0,3,4	1,2,5,6
8	$+$	$+$	0,1,4,5,8	2,3,6,7
	$-$	$+$	0,3,4,7,8	1,2,5,6

Table 2: The rank r of symmetric and anti-symmetric $\Gamma_{I_1\cdots I_r}C$

	η	ε	$s-t$ mod 8
Majorana	$+$	$-$	$1,2,8$
pseudo-Majorana	$+$	$+$	$6,7,8$
Majorana-Weyl	$-$	$+$	8
pseudo-Majorana-Weyl	$+$	$+$	8

Table 3: The values of $s-t$ where Majorana, pseudo-Majorana, Majorana-Weyl and pseudo-Majorana-Weyl spinors exist.

B Proof of Eq.(3.13)

In this appendix we show Eq.(3.13)

$$D-p-1=\frac{1}{2}n_f \tag{B.1}$$

follows from Eq.(3.10):

$$(\Gamma_I)^\alpha{}_\beta(\Gamma^{IJ_1\cdots J_{p-1}})^\gamma{}_\delta\Psi^{[a_1}_\beta\bar{\Psi}^{a_2}_\gamma\Psi^{a_3]}_\delta=0. \tag{B.2}$$

As in [19], when the fermion Ψ^a is a complex spinor we define a spinor Υ^a by

$$\Upsilon^a=\begin{pmatrix} P\Psi^a \\ \overline{P\Psi^a}^T \end{pmatrix}. \tag{B.3}$$

P is a chirality projection if Ψ is a Weyl spinor and the identity matrix otherwise. We define symmetric matrices Σ_I, $\tilde{\Sigma}_I$ by

$$\Sigma_I=\begin{pmatrix} 0 & \Gamma_I^T \\ \Gamma_I & 0 \end{pmatrix},\quad \tilde{\Sigma}_I=\begin{pmatrix} 0 & \Gamma_I \\ \Gamma_I^T & 0 \end{pmatrix}, \tag{B.4}$$

which satisfy

$$\tilde{\Sigma}_I \Sigma_J + \tilde{\Sigma}_J \Sigma_I = 2\eta_{IJ}. \tag{B.5}$$

We define

$$Z = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{B.6}$$

for $p = 1 \bmod 4$, $p = 2 \bmod 4$, $p = 3 \bmod 4$, $p = 4 \bmod 4$, respectively.

When Ψ^a is (pseudo-)Majorana(-Weyl) we define

$$\Sigma_I = \Gamma_I C, \quad \tilde{\Sigma}_I = C^{-1}\Gamma_I, \quad \Upsilon^a = P\Psi^a, \quad Z = \mathbb{1}. \tag{B.7}$$

Therefore Eq.(B.2) is equivalent to

$$\Sigma_I \Upsilon^{[a_1} \Upsilon^{a_2 T} Z \Sigma^{I J_1 \cdots J_{p-1}} \Upsilon^{a_3]} = 0, \tag{B.8}$$

where the square bracket denotes total anti-symmetrization in $(p+1)$-algebra indices and $\Sigma^{IJ_1\cdots J_{p-1}}$ is defined as

$$\begin{aligned} \Sigma^{IJ_1\cdots J_{p-1}} &= \Sigma^{[I}\tilde{\Sigma}^{J_1} \cdots \tilde{\Sigma}^{J_{p-2}}\Sigma^{J_{p-1}]} \quad \text{for odd } p, \\ \Sigma^{IJ_1\cdots J_{p-1}} &= \tilde{\Sigma}^{[I}\Sigma^{J_1} \cdots \tilde{\Sigma}^{J_{p-2}}\Sigma^{J_{p-1}]} \quad \text{for even } p, \end{aligned} \tag{B.9}$$

where the square bracket denotes total anti-symmetrization in the Lorentz indices. Since we have doubled the size of the spinors when the fermions Ψ are complex spinors, we can always go to a real basis by a similarity transformation. Therefore (B.8) is equivalent to

$$(\Sigma_I \mathbb{P})_{\alpha(\beta} (Z \Sigma^{IJ_1\cdots J_{p-1}} \mathbb{P})_{\gamma\delta)} = 0, \tag{B.10}$$

where for complex spinors

$$\mathbb{P} = \begin{pmatrix} P & 0 \\ 0 & \tilde{P}^T \end{pmatrix} \tag{B.11}$$

with $\tilde{P} = P$ for t even and $\tilde{P} = 1 - P$ for t odd for Weyl spinors and $P = \tilde{P} = 1$ otherwise, and $\mathbb{P} = P$ for (pseudo-)Majorana spinors. Contracting (B.10) with $(\tilde{\Sigma}^K)^{\beta\alpha}$ we obtain

$$n_f (Z \Sigma^{KJ_1\cdots J_{p-1}} \mathbb{P})_{\gamma\delta} + 2(Z \Sigma_{IJ_1\cdots J_{p-1}} \tilde{\Sigma}^K \Sigma^I \mathbb{P})_{\gamma\delta} = 0, \tag{B.12}$$

where n_f is the spinor size of fermions Ψ counted in real number. From (B.12) we obtain

$$(n_f - 2(D - p - 1))(Z \Sigma^{KJ_1\cdots J_{p-1}} \mathbb{P})_{\gamma\delta} = 0. \tag{B.13}$$

Thus we have obtained Eq.(B.1):

$$D - p - 1 = \frac{1}{2} n_f \tag{B.14}$$

as a necessary condition for Eq.(B.2) to vanish. One can check that it is also a sufficient condition.

The derivation of Eq.(5.7) is similar, the main difference is the spinor size n_θ of the space-time spinor field θ and the fact that in (B.10) only three spinor indices are symmetrized whereas in the case of super p-brane the closure of H leads to a condition

$$(\Sigma_I \mathbb{P})_{(\alpha\beta}(Z\Sigma^{IJ_1\cdots J_{p-1}}\mathbb{P})_{\gamma\delta)} = 0, \tag{B.15}$$

i.e. four spinor indices are symmetrized. From (B.15) one obtains [19]

$$D - p - 1 = \frac{1}{4} n_\theta, \tag{B.16}$$

for $2 < p+1 < D$. In all the cases listed in the Table 1, $n_f = n_{min}$ and $n_\theta = n_{min} \times \mathcal{N}_{min}$ with $\mathcal{N}_{min} = 2$. The difference of the factors $\frac{1}{2}$ and $\frac{1}{4}$ in (B.10) and (B.16) is a consequence of the fact that in (B.10) three spinor indices were symmetrized whereas in (B.15) four spinor indices were symmetrized. Since the supersymmetric reduced model actions are obtained after fixing the fermionic gauge symmetry of the super p-brane actions which reduces the degrees of freedom of θ by half, i.e. $n_f = \frac{1}{2} n_\theta$, this difference of the factors is what it should be.

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